

Hydrodynamics from the Dp-brane

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ABSTRACT: We compute the shear viscosity η , the bulk viscosity ζ , and the speed of sound v_s for theories dual to the near horizon geometries that arise from a stack of black Dp-branes for $p = 2, \dots, 6$ in the decoupling limit. The main new result is the bulk viscosity which, for all p , is found to be related to the speed of sound by the simple relation $\zeta/\eta = -2(v_s^2 - \frac{1}{p})$. For completeness the shear viscosity is rederived from gravitational perturbations in the shear and scalar channels. We comment on technical issues like the counterterms needed, or the possible dependence on the conformal frame.

KEYWORDS: Black Holes in String Theory, AdS-CFT Correspondence.

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1. Introduction and statement of results

The AdS/CFT correspondence is believed to apply to stacks of D p -branes for arbitrary p [1]. The non-conformality of the D p -brane backgrounds, being welcome for the physics, introduces a host of technical difficulties. On the side of the boundary theory, the identification as a “bona fide” QFT works only within some energy windows. Still a lot of physics has been extracted from such an effective description. For example, for $p > 3$, wrapping some internal world-volume directions of the brane along a small enough compact manifold has become an industry for modelling supersymmetric versions of QCD, starting with D4-branes in [2], D5-branes in [3] and D6-branes in [4].

The UV completion of these theories gives in most cases an elusive object. On the gravity side, this translates into the absence, so far, of a full fledged holographic renormalization program, as complete as the one developed for asymptotically AdS metrics (see [5] and references therein). In the case of D p -branes, the metric in the decoupling limit is only conformal to AdS $_{p+2}$. Still a minimally modified set of counterterms was proposed in [6] to renormalize the on-shell boundary action. This, by itself, sets the thermodynamics under control and allows for the computation of the energy-momentum tensor, which matches the one obtained from the asymptotically flat completion [7]. In this paper we will see that such counterterms are also enough to obtain the shear viscosity from the two point function of the energy-momentum tensor. A full construction of the renormalized action

is clearly beyond the scope of this note, and should presumably proceed along the lines investigated in [8].

In this paper we will study the transport coefficients of the dual plasma in the universal hydrodynamic regime. This implies that all time and length scales have to be very large as compared to the microscopic correlation lengths, which are set by the inverse temperature T^{-1} . Having control over the thermodynamics gives already information about the speed of sound $v_s^2 = \partial P / \partial \epsilon$. The following expression

$$v_s^2 = \frac{5 - p}{9 - p} \tag{1.1}$$

albeit evident from the form of the renormalized energy-momentum tensor [6, 7], was to our knowledge first written in [9]. It signals the onset of a tachyonic instability that in the dual field theory corresponds to the fact that for $p > 5$ the specific heat becomes negative [10].

Among other results in this paper, we will recover (1.1) from the pole structure of retarded correlators of the energy-momentum tensor. The implementation of this program in the context of AdS/CFT correspondence was initiated in [11, 12] and we will make use of the clean formulation advocated in [13] that neatly explains how to obtain the relevant dispersion relations from gauge invariant fluctuations of the supergravity fields. The key observation is the fact that the relevant boundary conditions for the fluctuations are the same as the so called quasi-normal modes in the context of black hole perturbation analysis. Quasinormal modes for p-branes have been studied in the past, albeit in different context. In [14, 15] the emphasis was on the thermalization properties of the dual plasma. In [16] the aim was to investigate the decay of probe-branes in a thermal AdS background.

In contrast, hydrodynamics is related to the long wavelength/frequency limit of perturbations, hence to the lowest such quasinormal modes. Symmetry analysis allows to catalog the fluctuations in three decoupled channels. In two of them, so called shear and sound channels, the general formalism predicts the appearance of poles of the following form

$$\text{shear channel} \rightarrow \omega = -\frac{i\eta}{\epsilon + P}q^2, \tag{1.2}$$

$$\text{sound channel} \rightarrow \omega = v_s q - i\frac{\eta}{\epsilon + P} \left(\frac{p-1}{p} + \frac{\zeta}{2\eta} \right) q^2 + \dots \tag{1.3}$$

Microscopically, such dispersion relations turn into poles of the retarded two point functions of certain components of the energy-momentum tensor. The relevant two point functions were precisely identified in [13] with fluctuations of the background metric that transform respectively as a vector and a scalar under the little group $SO(p-1)$ (resp. shear and sound channels). As we will show below, the dispersion relations allow to recover both the speed of sound given in (1.1) as well as the shear and bulk viscosities. The results are best expressed in terms of the following quotients

$$\frac{\eta}{s} = \frac{1}{4\pi}, \quad \frac{\zeta}{\eta} = \frac{2(3-p)^2}{p(9-p)}. \tag{1.4}$$

The famous equation on the left hand side was first obtained in [17] both in the context of the membrane paradigm and in the AdS/CFT formalism by relating the shear viscosity with

the diffusion of the R-current. Just for completeness, we add here a genuinely gravitational computation.

Also following an observation of [18] we notice that from equations (1.2) and (1.3) the relation

$$\frac{\zeta}{\eta} = -2 \left(v_s^2 - \frac{1}{p} \right) \tag{1.5}$$

holds exactly true for all values of p . We will have more to say about this equation in the conclusions.

The paper is organized as follows. In section 2 we shall establish the reduced model in the $p + 2$ dimensional bulk, and argue that it only contains a scalar field in addition to the metric. In the next section we shall examine the fluctuations and obtain the transport coefficients announced in this introduction. We add a short section which starts by raising the question about the correct choice of conformal frame. Unfortunately, the final results for the transport coefficients exhibit no dependence on the frame, and thus, shed no further light into the question. The paper closes with some concluding remarks and comparison with related results in the literature.

2. Consistent reduction and thermodynamics

In the Einstein frame, the relevant supergravity field profiles that correspond to the decoupling limit of a stack of Dp-branes read as follows

$$ds_{10}^2 = G_{MN}^{(10)} dx^M dx^N \tag{2.1}$$

$$= H^{-\frac{7-p}{8}}(r) (-f(r) dt^2 + dx_1^2 + \dots + dx_p^2) + H^{\frac{p+1}{8}}(r) \left(\frac{dr^2}{f(r)} + r^2 d\Omega_{8-p}^2 \right),$$

$$e^{\phi(r)} = H(r)^{\frac{3-p}{4}}, \tag{2.2}$$

$$F_{(8-p)} = \frac{7-p}{L} \omega_{S_{8-p}}, \tag{2.3}$$

where $H(r) = (L/r)^{7-p}$, $f(r) = 1 - (r_0/r)^{7-p}$, and $d\Omega_{8-p}^2$ stands for the metric of a $8 - p$ sphere of unit radius. This solution is obtained from a type II supergravity lagrangian where, keeping only the relevant degrees of freedoms, we have

$$S_{II} = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-G} \left[R(G) - \frac{1}{2} \partial_M \phi \partial^M \phi - \frac{1}{2n!} e^{a\phi} F_{(8-p)}^2 \right], \tag{2.4}$$

with $a = (3 - p)/2$ (i.e. we are considering magnetically charged branes). Consider the following ansatz for a dimensional reduction

$$ds_{10}^2 = e^{-\frac{2(8-p)}{p}B(r)} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{2B(r)} L^2 d\Omega_{8-p}^2 \tag{2.5}$$

$$= e^{-\frac{2(8-p)}{p}B(r)} \left(-c_T^2(r) dt^2 + c_X^2(r) \sum_{i=1}^p dx_i^2 + c_R^2(r) dr^2 \right) + e^{2B(r)} L^2 d\Omega_{8-p}^2,$$

where $g_{\mu\nu}$, $\mu, \nu = 0, 1, \dots, p$ stands for the metric in the $p + 2$ dimensional Einstein frame. Plugging this ansatz into the equations of motion derived from (2.4) one obtains a coupled

system of differential equations for the metric components $c_T(r), c_X(r)$ and $c_R(r)$, as well as for the dilaton $\phi(r)$ and the breathing mode $B(r)$. One can then check easily that the following identification yields a consistent reduction

$$B(r) = -\frac{3-p}{4(7-p)}\phi(r). \tag{2.6}$$

By this we mean that the equations of motion for $B(r)$ and $\phi(r)$, and perturbations thereof, become identical. Therefore, from here on we shall work within this truncation. The effective system in $p+2$ spacetime is governed by a system of equations that can be derived from the following action [19]

$$I_{\text{bulk}} = \frac{1}{16\pi G_{p+2}} \int d^{p+2}x \sqrt{-g} \left(R(g) - \frac{\beta}{2} \partial_\mu \phi \partial^\mu \phi - \mathcal{P}(\phi) \right), \tag{2.7}$$

with

$$\frac{1}{G_{p+2}} = \frac{2\pi^{\frac{9-p}{2}} L^{8-p}}{\Gamma\left(\frac{9-p}{2}\right) G_{10}},$$

where $\mathcal{P}(\phi)$ is the effective potential for the dilaton, and we have not bothered to normalize the field ϕ canonically

$$\mathcal{P}(\phi) = -\frac{(7-p)(p-9)}{2L^2} e^{\frac{4(3-p)}{p(7-p)}\phi(r)}, \quad \beta = \frac{8(9-p)}{p(7-p)^2}. \tag{2.8}$$

The effective equations of motion

$$R_{\mu\nu}(g) = \frac{\beta}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{p} g_{\mu\nu} \mathcal{P}(\phi), \tag{2.9}$$

$$\square \phi = \frac{\mathcal{P}'(\phi)}{\beta}, \tag{2.10}$$

are satisfied by the background profiles (setting $l_s = 1$)

$$c_T^2(r) = \left(\frac{r}{L}\right)^{\frac{9-p}{p}} f(r), \quad c_X^2(r) = \left(\frac{r}{L}\right)^{\frac{9-p}{p}}, \quad c_R^2(r) = \frac{1}{f(r)} \left(\frac{r}{L}\right)^{\frac{p^2-8p+9}{p}}, \tag{2.11}$$

as well as

$$\phi(r) = -\frac{(3-p)(7-p)}{4} \log\left(\frac{r}{L}\right), \tag{2.12}$$

from where the Hawking temperature and entropy density come straight

$$T = \frac{7-p}{4\pi r_0} \left(\frac{r_0}{L}\right)^{(7-p)/2}, \quad s = \frac{1}{4G_{p+2}} \left(\frac{r_0}{L}\right)^{(9-p)/2}. \tag{2.13}$$

If desired, using the AdS/CFT dictionary, it is straightforward to translate the entropy density into field theoretical quantities, involving the rank N , the temperature T , and the gauge coupling $\lambda = N g_{\text{YM}}^2 = (2\pi)^{p-2} g_s l_s^{p-3}$

$$s \sim N^2 \lambda^{-\frac{3-p}{5-p}} T^{\frac{9-p}{5-p}}. \tag{2.14}$$

3. Fluctuations in the hydrodynamic regime

Let us consider fluctuations of the bulk fields $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$, $\phi \rightarrow \phi + \delta\phi$, and focus on a single Fourier component that propagates along the coordinate $z = x^p$

$$\delta g_{\mu\nu}(t, z, r) = e^{-i(\omega t - qz)} h_{\mu\nu}(r), \quad (3.1)$$

$$\delta\phi(t, z, r) = e^{-i(\omega t - qz)} \varphi(r). \quad (3.2)$$

Standard analysis proceeds by grouping the fluctuations into three irreducible channels according to their helicity s under the little group $\text{SO}(p-1)$ [11]

$$s = 0 \rightarrow \text{sound channel} : h_{tt}, h_{tz}, h_{zz}, h_{rr}, h_{tr}, h_{zr}, h, \varphi \quad (3.3)$$

$$s = 1 \rightarrow \text{shear channel} : h_{ta}, h_{za}, h_{ra} \quad (3.4)$$

$$s = 2 \rightarrow \text{scalar channel} : h_{ab} - \delta_{ab} \frac{h}{p-1} \quad (3.5)$$

with $a, b = 1, \dots, p-1$ and $h = \sum_a h_{aa}$. Let us parametrize fluctuations as usual with $H_{\mu\nu}(r)$ such as

$$h_{tt}(r) = c_T^2 H_{tt}(r), \quad (3.6)$$

$$h_{\mu j}(r) = c_X^2 H_{\mu j}(r), \quad (3.7)$$

with $h_{j\mu}(r) = h_{\mu j}(r)$ and $j = 1, 2, \dots, p$. We have fixed coordinates such that $\delta g_{\mu r} = 0$. This leaves still a residual gauge freedom under the infinitesimal diffeomorphisms $x^\mu \rightarrow x^\mu + \xi^\mu$, $\delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$ and $\delta\phi \rightarrow \delta\phi - \partial^\mu \phi \xi_\mu$ with $\xi_\mu = \xi_\mu(r) e^{-i\omega t + iqz}$ and covariant derivatives taken with respect to the background metric. Rather than fixing completely the gauge, it is more convenient to switch over to a set of gauge invariant fluctuations [13]

$$\begin{cases} Z_0 = q^2 \frac{c_T^2}{c_X^2} H_{tt} + 2q\omega H_{tz} + \omega^2 H_{zz} + \left(q^2 \frac{\ln'(c_T)}{\ln'(c_X)} \frac{c_T^2}{c_X^2} - \omega^2 \right) H, \\ Z_\varphi = \varphi - \frac{\phi'}{\ln'(c_X^{2(p-1)})} H_{aa}, \end{cases} \quad (3.8)$$

$$Z_1 = qH_{ta} + \omega H_{za}, \quad (3.9)$$

$$Z_2 = H_{ab}, \quad (3.10)$$

where $H = \frac{1}{p-1} \sum_a H_{aa}$. The ODEs obeyed by the fluctuations $H_{\mu\nu}$ can be found in appendix A. Here we just present the equations for the gauge invariant fluctuations. The following dimensionless ratios are natural in order to examine the hydrodynamic regime

$$\mathfrak{w} = \frac{\omega}{2\pi T}, \quad \mathfrak{q} = \frac{q}{2\pi T}. \quad (3.11)$$

In each one of the three channels we shall obtain decoupled second order differential equations that will be solved in the nontrivial lowest order limit when $\mathfrak{w} \ll 1$ and $\mathfrak{q} \ll 1$ with $\mathfrak{w}/\mathfrak{q} = \lambda(\mathfrak{q})$, where $\lambda(\mathfrak{q})$ is a function of \mathfrak{q} analytic at $\mathfrak{q} \rightarrow 0$. The analysis of the

characteristic exponents near r_0 allows us to parametrize our gauge invariant functions as follows

$$Z_x(r) = f(r)^{-i\frac{\mathfrak{w}}{2}} Y_x(r), \quad (3.12)$$

with $Y_x(r)$ analytic at $r = r_0$. In this way we are selecting ingoing boundary conditions at the horizon. Then we only have to solve perturbatively for $Y_x(r)$ in the hydrodynamic limit. The dispersion relation is obtained from imposing Dirichlet boundary conditions [13]

$$Z_x(r)|_{r=\infty} = 0. \quad (3.13)$$

3.1 Shear channel

Taking a suitable combination of the three equations in section A.2 one obtains the ODE satisfied by the gauge independent vector fluctuation Z_1

$$Z_1'' + \left[\left(\frac{q^2}{\omega^2} \frac{c_T^2}{c_X^2} - 1 \right)^{-1} \ln' \left(\frac{c_X^2}{c_T^2} \right) + \ln' \left(\frac{c_X^{p+2}}{c_T c_R} \right) \right] Z_1' + c_R^2 \left(\frac{\omega^2}{c_T^2} - \frac{q^2}{c_X^2} \right) Z_1 = 0. \quad (3.14)$$

Plugging the ingoing ansatz (3.12) we obtain an equation for $Y_1(r)$ which can be solved perturbatively giving in all cases

$$Z_1(r) = C_V f(r)^{-i\frac{\mathfrak{w}}{2}} \left(1 + i \frac{q^2}{2\mathfrak{w}} f(r) + \mathcal{O}(\mathfrak{w}, q^2) \right), \quad (3.15)$$

where C_V is an unimportant normalization factor. From here and (3.13) the familiar dispersion relation follows

$$\mathfrak{w} = -i \frac{q^2}{2}. \quad (3.16)$$

Restoring ω and q , and comparing with (1.2), gives the expected universal result for η/s shown in the left hand equation of (1.4).

3.2 Sound channel

In this channel we end up with two scalar fluctuations Z_φ and Z_0 . They satisfy the following equations

$$Z_\varphi'' + \ln' \left(\frac{c_T c_X^p}{c_R} \right) Z_\varphi' + c_R^2 \left[\frac{\omega^2}{c_T^2} - \frac{q^2}{c_X^2} - \frac{2(3-p)}{p} \left(\frac{3-p}{9-p} + \frac{\phi'}{\ln'(c_X)} \frac{2}{(7-p)p} \right) \mathcal{P} \right] Z_\varphi = 0 \quad (3.17)$$

$$Z_0'' + \mathcal{F}(r) Z_0' + \mathcal{G}(r) Z_0 + \mathcal{H}(r) Z_\varphi = 0, \quad (3.18)$$

modulo the background equations of motion. The coefficients in (3.18) are given by

$$\mathcal{F}(r) = \ln' \left(\frac{c_T c_X^p}{c_R} \right) - 4 \ln' \left(\frac{c_T}{c_X} \right) + \xi(r), \quad (3.19)$$

$$\mathcal{G}(r) = c_R^2 \left(\frac{\omega^2}{c_T^2} - \frac{q^2}{c_X^2} \right) + 4 \left[\ln' \left(\frac{c_T}{c_X} \right) \right]^2 - \ln' \left(\frac{c_T}{c_X} \right) \xi(r), \quad (3.20)$$

$$\mathcal{H}(r) = 8 \frac{q^2}{\omega^2} \frac{(3-p)}{p(p-7)} \frac{c_T}{c_X^2} (c_T' - c_T \ln'(c_X)) \xi(r), \quad (3.21)$$

with

$$\xi(r) = \frac{q^2 (c_T^2)' \frac{\ln''(c_X)}{(\ln'(c_X))^2} \left(1 - \frac{\ln''(c_T) \ln'(c_X)}{\ln''(c_X) \ln'(c_T)}\right) + 2\omega^2 p (c_X^2)' \left(1 - \frac{\ln'(c_T)}{\ln'(c_X)}\right)}{q^2 c_T^2 \left(\frac{\ln'(c_T)}{\ln'(c_X)} + p - 1\right) - \omega^2 p c_X^2}. \quad (3.22)$$

Plugging as before the ingoing ansatz (3.12) and solving perturbatively one finds that the only non-singular solution to (3.17) is a constant, which we set to zero by the boundary conditions at infinity. Inserting now $Z_\varphi = 0$ into (3.18) and solving perturbatively for Z_0 gives

$$Z_0(r) = C_S f(r)^{-i\frac{\mathfrak{w}}{2}} \left(1 - \frac{(1 + i4\mathfrak{w})q^2 f(r)}{(7-p)q^2 - (9-p)\mathfrak{w}^2} + \mathcal{O}(\mathfrak{w}^2, q^2, \mathfrak{w}q)\right). \quad (3.23)$$

with C_S a normalization constant. Imposing the Dirichlet boundary condition (3.13) gives an expression for $\mathfrak{w}(q)$ that can be expanded as follows

$$\mathfrak{w} = \sqrt{\frac{5-p}{9-p}} q - i \frac{2}{9-p} q^2 + \dots \quad (3.24)$$

Comparing this expression with the dispersion relation (1.3), and using (1.4) we identify finally

$$v_s^2 = \frac{5-p}{9-p}, \quad \frac{\zeta}{\eta} = \frac{2(3-p)^2}{p(9-p)}, \quad (3.25)$$

as claimed in the introduction.

3.3 Scalar channel

In this subsection, and just for the sake of completeness, we reobtain the shear viscosity through the Kubo formula. It is little more than an academic exercise, given the general theorem [20]. However the fact that the metric is not asymptotically AdS makes it worth to explore this in detail. As usual, the equation satisfied by $Z_2 = H_{ab}$ is that of a minimally coupled scalar

$$Z_2'' + \ln' \left(\frac{c_T c_X^p}{c_R}\right) Z_2' + c_R^2 \left(\frac{\omega^2}{c_T^2} - \frac{q^2}{c_X^2}\right) Z_2 = 0. \quad (3.26)$$

In the hydrodynamic limit the ingoing solution to (3.26) exhibits no poles and can be expanded as follows

$$Z_2(\omega, r) = f(r)^{-i\frac{\mathfrak{w}}{2}} (1 + \mathcal{O}(\mathfrak{w}^2, q^2)). \quad (3.27)$$

In this case the standard roundabout invokes the Kubo formula

$$\eta = - \lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im} G_R(\omega), \quad (3.28)$$

with G_R the retarded correlator of the relevant components of the energy-momentum tensor

$$G_R(\omega) = -i \int dt d^p x e^{i\omega t} \theta(t) \langle [T_{xy}(t, \vec{x}), T_{xy}(0, \vec{0})] \rangle. \quad (3.29)$$

The evaluation of the retarded correlator calls for the expansion of the renormalized boundary action up to second order in the fluctuation $H_{\mu\nu}$. Whereas such an object has been rigorously constructed for actions possessing asymptotically locally AdS_d backgrounds (see [5])

and references therein), for the case of Dp-branes only partial results are known. In [6] appropriate counterterms were found on a case by case basis that properly renormalized the action, giving a renormalized energy-momentum tensor. One can easily see that these counterterms, with the correct coefficients, are exactly reproduced by the general expression given in [21] which we follow here. Let us express the regularized action as

$$I = \sum_A I_A = I_{\text{bulk}} + I_{GH} + I_{ct}, \quad (3.30)$$

where I_{bulk} is as in (2.7), and

$$I_{GH} = \frac{1}{16\pi G_{p+2}} \int_{r=r_\infty} d^{p+1}x \sqrt{h} 2K, \quad I_{ct} = \frac{1}{16\pi G_{p+2}} \int_{r=r_\infty} d^{p+1}x \sqrt{h} (2\mathcal{W}(\phi) + \dots). \quad (3.31)$$

$\mathcal{W}(\phi)$ is the superpotential, related to the potential $\mathcal{P}(\phi)$ by the non-linear equation

$$\mathcal{P}(\phi) = \frac{2}{\beta} (\partial_\phi \mathcal{W}(\phi))^2 - \frac{p+1}{p} \mathcal{W}(\phi)^2, \quad (3.32)$$

whose solution, for $\mathcal{P}(\phi)$ as in (2.8), is given by

$$\mathcal{W}(\phi) = \frac{(9-p)}{2L} e^{\frac{2(3-p)}{p(7-p)}\phi(r)}. \quad (3.33)$$

The dots in (3.31) denote higher curvature invariants on the induced hypersurface. After expanding I to second order in the (purely time dependent) perturbation

$$h^a_b(t, r) = \int \frac{d\omega}{2\pi} e^{-i\omega t} f(\omega) Z_2(\omega, r), \quad (3.34)$$

we can cast all contributions in the form of boundary terms $\delta I = \sum_A \delta I_A$ with

$$\delta I_A = \int d^p x \frac{d\omega}{2\pi} f(\omega) f(-\omega) \mathcal{F}_A(\omega, r) \Big|_{r_0}^\infty \quad (3.35)$$

and find thereafter

$$\begin{aligned} \mathcal{F}_{\text{bulk}} &= \frac{1}{16\pi G_{p+2}} \left(\frac{r_0^{7-p}}{L^{8-p}} \right) \left(-\frac{9-p}{2p} \left(\frac{r}{r_0} \right)^{7-p} + \frac{9-p}{2p} + i \frac{3(7-p)}{4} \mathfrak{w} + \dots \right), \\ \mathcal{F}_{GH} &= \frac{1}{16\pi G_{p+2}} \left(\frac{r_0^{7-p}}{L^{8-p}} \right) \left(\frac{(9-p)(p+1)}{2p} \left(\frac{r}{r_0} \right)^{7-p} - \frac{9+p}{2p} - i(7-p)\mathfrak{w} + \dots \right), \\ \mathcal{F}_{ct} &= \frac{1}{16\pi G_{p+2}} \left(\frac{r_0^{7-p}}{L^{8-p}} \right) \left(-\frac{9-p}{2} \left(\frac{r}{r_0} \right)^{7-p} + \frac{9-p}{4} + \dots \right), \end{aligned}$$

where the dots stand for terms of $\mathcal{O}(r_0/r, \mathfrak{w}^2)$. Adding up and using the Minkowskian prescription of [22] we obtain the retarded correlator

$$G_R(\omega) = 2\mathcal{F}(r)|_{r=\infty} = \frac{1}{16\pi G_{p+2}} \left(\frac{r_0^{7-p}}{L^{8-p}} \right) \left(\frac{5-p}{2} - i \frac{7-p}{2} \mathfrak{w} \right). \quad (3.36)$$

We see that the counterterm contributes to the real part of the renormalized correlator¹. From (3.36) the shear viscosity can be extracted as usual by means of Kubo formula

$$\eta = \frac{1}{16\pi^2 G_{p+2}} \frac{7-p}{4} \frac{r_0^{7-p}}{TL^{8-p}} \tag{3.37}$$

and, using (2.13), again the well known result $\eta/s = 1/4\pi$ is recovered.

4. Frame (in)dependence

In the AdS/CFT correspondence, one delicate issue concerns the correct identification of the bulk field perturbation that couples correctly to the desired boundary operator. We are interested in perturbing bulk fields that couple exactly to the boundary energy-momentum tensor. In asymptotically AdS spaces, the energy-momentum tensor couples naturally to the bulk metric. However, outside the well tested arena of such backgrounds, we are on less firm grounds. In the case of Dp-branes, the metric is asymptotically conformally AdS_{p+2} in the Einstein frame². The conformal factor that asymptotically deviates its profile from AdS is given by an appropriate power of the function $e^{\phi(r)}$ where $\phi(r)$ is the dilaton. Hence we may consider a family of conformally related metrics, parametrized by $\alpha \in \mathbb{R}$ as follows

$$g_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu}^{(\alpha)}. \tag{4.1}$$

Clearly $g_{\mu\nu}^{(0)}$ is the Einstein frame metric whose background value is given in (2.11). Another important case yields the so called “dual frame” [19], and is obtained by tuning α to the following value

$$\alpha_D = -\frac{2(3-p)}{p(7-p)},$$

which sets exactly $g_{\mu\nu}^{(\alpha_D)}$ to the following asymptotically AdS_{p+2} black hole metric

$$ds_{(\alpha_D)}^2 = \left(\frac{r}{L}\right)^{5-p} (-f(r)dt^2 + d\vec{x}_p d\vec{x}_p) + \left(\frac{L}{r}\right)^2 \frac{dr^2}{f(r)}. \tag{4.2}$$

In [19], this conformal frame was argued to yield the natural “holographic” bulk metric, where the AdS/CFT duality should work most transparently. Notice that, in principle, perturbation of the metric in different frames would couple to different combinations of the energy-momentum tensor and the “glueball operator” in the boundary field theory. Until the question among the dual frame and the Einstein frame is settled, the natural way to proceed is to see if indeed the results depend upon the choice of such frame. We can repeat the analysis of the paper in terms of the pair $(g_{\mu\nu}^{(\alpha)}, \phi)$ for an arbitrary α . In particular

¹Such contribution is essential in order to fulfill Ward identities, and is typically missed in non-covariant treatments of the counterterm action. It does not contribute to the coefficient of ω but, to our knowledge, this is not a general statement when higher curvature counterterms are included.

²Except for $p = 5$ where it asymptotes to flat $M^{1,6}$

this amounts to replacing $g_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu}^{(\alpha)}$ in equations (2.9) and (2.10) . Introducing now perturbations as follows

$$g_{\mu\nu}^{(\alpha)} \rightarrow g_{\mu\nu}^{(\alpha)} + \delta g_{\mu\nu}^{(\alpha)}, \tag{4.3}$$

$$\phi \rightarrow \phi + \delta\phi, \tag{4.4}$$

all the intermediate equations acquire a dependence on α . For example, instead of (3.17) one gets

$$Z''_{\varphi} + \left(\ln' \left(e^{p\alpha\phi} \frac{c_T c_X^p}{c_R} \right) \right) Z'_{\varphi} + c_R^2 \left[\frac{\omega^2}{c_T^2} - \frac{q^2}{c_X^2} - \frac{2(3-p)}{p} \left(\frac{3-p}{9-p} + \frac{\phi'}{\ln'(c_X)} \left(\frac{2}{(7-p)p} + \alpha \frac{3-p}{9-p} \right) \right) \mathcal{P} \right] Z_{\varphi} = 0 \tag{4.5}$$

and so on. The analysis can be carried along the same lines as in before, and in the final result all the α dependence cancels out exactly. Stated precisely, the expressions given in (1.1) and (1.4) are frame independent.

5. Conclusions

In this note, we have completed the table of fluid transport coefficients of the non-abelian quantum plasmas that are dual to the gravitational background of a stack of non-extremal Dp-branes in the decoupling limit for $p = 2, \dots, 6$. We have recovered known values for the speed of sound (1.1) and the quotient of the shear viscosity over the entropy (1.4) from poles of energy-momentum tensor correlators, as well as from the Kubo formula. The main new result is the expression for the bulk viscosity given in (1.4), which leads to the compact relation (1.5). Besides we have clarified some aspects related to the holographic renormalization and the frame dependence of the metric.

Let us comment and compare with partial results obtained in the literature in similar contexts. In [23], Parnachev and Starinets also investigate the hydrodynamic properties of thermal “little string theory” (LST), which is dual to a stack of black NS5-branes. Their results can be seen to agree with ours for $p = 5$, reflecting the fact that viscosity (as it happens with the entropy) is an S-duality invariant.

In reference [18], Benincasa and Buchel consider the background of a stack of D4-branes with one compactified dimension. This seemingly complicated geometry led them to introduce up to three independent scalar modes. Finally the dispersion relation they obtain matches precisely our eq. (3.24) with $p = 4$ ³. The disagreement comes from extracting the

³This points to a manifestation of the D4 structure that underlies the construction. However this fact is less than trivial. One can look at the equations for the fluctuations in [18], and find that they agree with our equations (both for $H_{\mu\nu}$ in the appendix A, and for Z_x in the main text) by setting instead $p = 3$. Also the gauge invariant combinations are given by (3.8) with $p = 3$. This happens because these expressions are only sensitive to the value of p that appears in the metric ansatz. It is only upon inserting the precise values of the field profiles, that the authors of [18] use the ones in (2.1) and (2.2) with $p = 4$. Somehow the p in the final dispersion relation (3.24) refers to its value in the background profile and loses track of the form of the metric ansatz.

bulk viscosity, where the authors of [18], having in mind a three dimensional fluid, use a parametrization of the dispersion relation which is precisely (1.3) with $p = 3$. The obtained value of the bulk viscosity satisfies a relation with the speed of sound which is (1.5) with $p = 3$ instead of $p = 4$. Intrigued by this mismatch we discovered an identity that extends our equation (1.5) and encompasses also (1.4) of [18]. Namely, one can replace $p \rightarrow d$ in (1.3) and solve again for v_s and ζ/η by comparison with the dispersion relation (3.24) while keeping p and d independent

$$v_s^2 = \frac{5-p}{9-p}, \quad \frac{\zeta}{\eta} = \frac{8d - 2(9-p)(d-1)}{d(9-p)}. \quad (5.1)$$

With this, one can verify that the following identity holds for any p and d

$$\frac{\zeta}{\eta} = -2 \left(v_s^2 - \frac{1}{d} \right). \quad (5.2)$$

Whether this can be ascribed a meaning or happens to be an arithmetic coincidence we don't know yet. Anyway, as announced, the results of [18] are recovered exactly by setting $d = 3$ and $p = 4$ in (5.1) and (5.2). Obviously this extension is motivated by the possibility of defining lower dimensional fluids from higher dimensional UV-field theories (e.g. via Kaluza-Klein compactification), but notice that it holds as well for $d > p$. Anyway, the persistence of an analytic pattern like (1.5), or its generalization (5.2), looks extremely appealing. The numerical prefactor -2 is not universal (~ -5 in the background dual to $N = 2^*$, [25]), but is the same as in the cascading theory [24]. This unexpected coincidence for a large amount of different backgrounds certainly deserves further study.

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A. Equations for the fluctuations

Let us give for completeness some of the intermediate equations that were skipped in the main body of the text for the sake of clarity.

A.1 Sound channel

Here we find a set of 5 second order equations for the fluctuations that enter the gauge

invariant expressions given in (3.8)

$$H''_{tt} + \ln' \left(\frac{c_T^2 c_X^p}{c_R} \right) H'_{tt} - \ln'(c_T) H'_{ii} \tag{A.1}$$

$$- c_R^2 \left(\frac{\omega^2}{c_T^2} H_{ii} + \frac{q^2}{c_X^2} H_{tt} + 2 \frac{q\omega}{c_T^2} H_{tz} \right) - \frac{2}{p} c_R^2 \frac{\partial \mathcal{P}}{\partial \phi} \varphi = 0 \tag{A.2}$$

$$H''_{tz} + \ln' \left(\frac{c_X^{p+2}}{c_T c_R} \right) H'_{tz} + \frac{c_R^2}{c_X^2} q \omega H_{aa} = 0 \tag{A.3}$$

$$H''_{zz} + \ln' \left(\frac{c_T c_X^{p+1}}{c_R} \right) H'_{zz} + \ln'(c_X) (H'_{aa} - H'_{tt}) + c_R^2 \left(\frac{\omega^2}{c_T^2} H_{zz} + 2 \frac{q\omega}{c_T^2} H_{tz} + \frac{q^2}{c_X^2} (H_{tt} - H_{aa}) \right) + \frac{2}{p} c_R^2 \frac{\partial \mathcal{P}}{\partial \phi} \varphi = 0 \tag{A.4}$$

$$H''_{aa} + \ln' \left(\frac{c_T c_X^{2p-1}}{c_R} \right) H'_{aa} + \ln'(c_X^{p-1}) (H'_{zz} - H'_{tt}) + c_R^2 \left(\frac{\omega^2}{c_T^2} - \frac{q^2}{c_X^2} \right) H_{aa} + \frac{2(p-1)}{p} c_R^2 \frac{\partial \mathcal{P}}{\partial \phi} \varphi = 0. \tag{A.5}$$

$$\varphi'' + \ln' \left(\frac{c_T c_X^p}{c_R} \right) \varphi' + c_R^2 \left(\frac{\omega}{c_T^2} - \frac{q^2}{c_X^2} \right) \varphi + \frac{1}{2} \phi' (H'_{ii} - H'_{tt})' - \frac{1}{\beta} \frac{\partial^2 \mathcal{P}}{\partial \phi^2} \varphi = 0 \tag{A.6}$$

Additionally there are three constraints associated with the gauge fixing condition $h_{\mu r} = 0$

$$H'_{ii} + \ln' \left(\frac{c_X}{c_T} \right) H_{ii} + \frac{q}{\omega} H'_{tz} + 2 \frac{q}{\omega} \ln' \left(\frac{c_X}{c_T} \right) H_{tz} + \beta \phi' \varphi = 0 \tag{A.7}$$

$$H'_{tt} - \ln' \left(\frac{c_X}{c_T} \right) H_{tt} + \frac{\omega}{q} \frac{c_X^2}{c_T^2} H'_{tz} - H'_{aa} - \beta \phi' \varphi = 0 \tag{A.8}$$

$$\ln'(c_T c_X^{p-1}) H'_{ii} - \ln'(c_X^p) H'_{tt} + c_R^2 \left(\frac{\omega^2}{c_T^2} H_{ii} + 2 \frac{q\omega}{c_T^2} H_{tz} + \frac{q^2}{c_X^2} (H_{tt} - H_{aa}) \right) - \beta \phi' \varphi' + c_R^2 \frac{\partial \mathcal{P}}{\partial \phi} \varphi = 0 \tag{A.9}$$

It is straightforward to check that together with equations (A.2)-(A.6) this system of 8 equations is not overdetermined, and one can construct easily 3 linear combinations that vanish identically “on shell” (that is, modulo the equations of motion).

A.2 Shear channel

Here we obtain two second order equations and one constraint

$$H''_{ta} + \ln' \left(\frac{c_X^{p+2}}{c_T c_R} \right) H'_{ta} - q \frac{c_R^2}{c_X^2} (q H_{tz} + \omega H_{za}) = 0, \tag{A.10}$$

$$H''_{za} + \ln' \left(\frac{c_T c_X^p}{c_R} \right) H'_{za} + \omega \frac{c_R^2}{c_T^2} (q H_{tz} + \omega H_{za}) = 0, \tag{A.11}$$

$$q H'_{za} + \omega \frac{c_X^2}{c_T^2} H'_{ta} = 0. \tag{A.12}$$

which are again differentially linearly dependent.

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